Journal of Experimental Psychology:
Human Perception and Performance
2007, Vol. 33, No. 6, 1410–1419

0096-1523/07/$12.00 DOI: 10.1037/0096-1523.33.6.1410

The Mental Representation of Numerical Fractions: Real or Integer?

Mario Bonato, Sara Fabbri, Carlo Umiltà, and Marco Zorzi
University of Padova

Numerical fractions are commonly used to express ratios and proportions (i.e., real numbers), but little is known about how they are mentally represented and processed by skilled adults. Four experiments employed comparison tasks to investigate the distance effect and the effect of the spatial numerical association of response codes (SNARC) for fractions. Results showed that fractions were processed componentially and that the real numerical value of the fraction was not accessed, indicating that processing the fraction's magnitude is not automatic. In contrast, responses were influenced by the numerical magnitude of the components and reflected the simple comparison between numerators, denominators, and reference, depending on the strategy adopted. Strategies were used even by highly skilled participants and were flexibly adapted to the specific experimental context. In line with results on the whole number bias in children, these findings suggest that the understanding of fractions is rooted in the ability to represent discrete numerosities (i.e., integers) rather than real numbers and that the well-known difficulties of children in mastering fractions are circumvented by skilled adults through a flexible use of strategies based on the integer components.

Keywords: fractions, magnitude representation, SNARC effect, distance effect, whole number bias

Fractions, denoted by the ratio between two integer numbers, indicate numerical quantities that correspond to real numbers and are a numerical format that is used in many circumstances (e.g., proportions and noninteger numbers). Indeed, the acquisition of rules for dealing with fractions is an important aspect of mathematical education. The most straightforward way for learning and processing fractions should be to access a mental representation of the fraction's numerical magnitude, that is, its real value. This would allow people to easily categorize, for example, 1/9 as smaller than 1/5. It is widely assumed that numerical magnitude is represented on the continuum of real numbers, conceived of as an analogical mental number line, and that a preverbal system of analogue magnitudes provides the foundations of human numerical and mathematical thinking (e.g., Dehaene, Dehaene-Lambertz, & Cohen, 1998; Gallistel & Gelman, 2000). Indeed, as clearly stated by Gallistel and Gelman, learning a system based on integer numbers would be rooted in the magnitude-based system (i.e., real numbers). Thus, if numerical magnitude were represented on the continuum of real numbers, then the mental representation of fractions should pose no challenge to the learner.

However, it is well known that children find fractions very hard to learn (e.g., Bright, Behr, Post, & Wachsmuth, 1988; Hartnett & Gelman, 1998; Mack, 1995; Smith, Solomon, & Carey, 2005). The difficulty in learning the concept of fraction seems to be related both to stepping away from the magnitude conveyed by each of the operands and to learning that a fraction's value can be bigger or smaller than the unit, depending on the ratio value (Stafylidou & Vosniadou, 2004). Many studies have reported these difficulties and suggested various teaching methods (e.g., Fuson & Abrahamson, 2005). It is widely agreed that children's difficulty with fractions is associated with their whole number (i.e., integer) knowledge, which represents numbers discretely and may therefore interfere with children's construction of the concept of fraction and rational numbers that are continuous (Ni & Zhou, 2005, for a review). Indeed, even after instruction, children make errors that are typically whole number intrusions. For example, they say that 1/56 is smaller than 1/75 because 56 is smaller than 75 (Smith et al., 2005). Mack (1995) described students explaining 3/5 as three whole objects cut into five pieces. Young elementary school aged children misunderstand fractional notation. For example, Smith et al. (2005) found that most third and fourth graders could not order fractions and could not explain why there are two numbers in a given fraction; moreover, children did not understand the density and infinite divisibility of number (many denied that there are any numbers between 0 and 1).

The robust tendency in using the single-unit counting scheme to interpret fractions and the difficulty in perceiving whole numbers as decomposable units has been referred to as whole number bias (Ni & Zhou, 2005). Because the concept of number for children in preschool and in the early elementary grades is constituted by relations and operations among positive integers, mastering fractions and rational numbers would seem to require a conceptual change (Hartnett & Gelman, 1998; Smith et al., 2005; but see Mix, Levine, & Huttenlocher, 1999, for a different view). Smith et al. suggested that such conceptual change involves two-way mappings between the domains of number and physical quantity, with

Mario Bonato, Sara Fabbri, Carlo Umiltà, and Marco Zorzi, Dipartimento di Psicologia Generale, University of Padova, Italy.

Sara Fabbri is now at CIMeC (Centro Interdipartimentale Mente/Cervello; Center for Mind/Brain Sciences), University of Trento, Italy.

This study was supported by MUR (Ministero dell'Università e della Ricerca; Italian Ministry of University and Research) PRIN 2003 grant to Carlo Umiltà and PRIN 2004 grant to Marco Zorzi and by a European Commission, Marie Curie Research Training Network, Numeracy and Brain Development grant to Marco Zorzi.

Correspondence concerning this article should be addressed to Marco Zorzi, Dipartimento di Psicologia Generale, University of Padova, via Venezia 8, 35131, Padova, Italy. E-mail: marco.zorzi@unipd.it

crucial support from the understanding of physical quantity as continuous and infinitely divisible.

Whatever be the developmental and learning processes that lead to the concept of rational number, the implicit assumption is that the difficulty in mastering fractions (whole number bias) is specific to children and would not be an issue for educated adults. However, experimental studies exploring the processing of fractions in skilled adults are lacking. If adults can process a fraction as a whole, its numerical magnitude should be directly and readily available in terms of its real value; this would index access to numerical magnitudes represented on the continuum of real numbers (Gallistel & Gelman, 2000). In fact, studies of decision making by Gigerenzer and colleagues (e.g., Gigerenzer & Hoffrage, 1999) suggest that fractions and proportions are hard to understand even for adults. One reason for this, as Gigerenzer and Hoffrage (1999) suggest, is that "humans seem developmentally and evolutionarily prepared to handle natural frequencies" (p. 430) but not proportions. That is to say, we find it much easier to think in terms of discrete numerosities than in terms of fractions, proportions, or rates (Butterworth, 2001; Zorzi & Butterworth, 1997; Zorzi, Stoianov, Becker, Umiltà, & Butterworth, 2006). Thus, representing the meaning of a fraction in terms of the numerosities of the numerator and of the denominator implies that the real value of the fraction is not readily accessible. Skilled adults might therefore circumvent this problem through the use of strategies that rely on processing of the integer components. A strong influence of the size of numerator and denominator would reveal componential processing and, thus, a reliance on a system based on exact, integer numbers even for a stimulus that is intrinsically noninteger (i.e., a whole number bias).

Thus, investigating how fractions are processed by skilled adults is interesting in its own right because no previous studies have looked at this issue, but it may also offer an insight into the relation between real numbers and integer numbers. The present investigation was built upon two classical effects in numerical cognition: the distance effect (Moyer & Landauer, 1967) and the spatial numerical association of response codes (SNARC) effect (Dehaene, Bossini & Giraux, 1993).

The distance effect (Moyer & Landauer, 1967) consists in the increase of reaction times (RTs) in a comparison task with decreasing distance between the target and a reference number. RTs are slower when the distance is small (e.g., 5 vs. 6) and become faster when the distance increases (e.g., 2 vs. 6). The effect is robust and widely used in studies of numerical cognition; it has been described not only for digits but also for patterns of dots and has also been described in animals (for a review, see Dehaene et al., 1998). Thus, when the task requires a comparison between numbers, the distance effect indicates that magnitudes are compared.

Magnitude comparison between two fractions should thus produce the typical distance effect. That is, if the magnitude of each fraction is directly accessed and compared, the distance effect should reflect the numerical distance between the corresponding real numbers. In contrast, componential processing of the two fractions should produce a distance effect that is related to the numerical magnitude of the numerator and denominator, separately. The former outcome would indicate a magnitude comparison between fractions, whereas the latter outcome would indicate a magnitude comparison between components of the fractions (i.e., a whole number bias).

The SNARC effect consists in the association between left responses and small numerical quantities and between right responses and large numerical quantities (Dehaene et al., 1993; Fias & Fischer, 2005, for a review). In a parity judgment task, a small number (e.g., 2, if we consider the 1–9 interval) is responded to faster with the left hand than with the right hand, whereas the opposite is true for a larger number (e.g., 8). This finding suggests that numbers are represented on a mental number line, spatially oriented from left to right (also see Zorzi, Priftis & Umiltà, 2002). Because the SNARC effect has been described in tasks in which the magnitude should in principle not be activated, such as parity judgment and phoneme monitoring (Fias, Brysbaert, Geypens & d'Ydewalle, 1996), its occurrence has been widely taken as evidence that numerical magnitude is automatically activated.

The presence of the SNARC effect when responding to a fraction should, in principle, reflect an interaction between the response side and the numerical magnitude of the fraction. In contrast, componential processing might produce instead an interaction between the response side and the numerical magnitude of the numerator and/or denominator. Accordingly, the aim of the present study was to address the question of whether a mental representation of the fraction's magnitude (i.e., a real number), independent of the magnitude of the operands, is formed and used by skilled adults. In other words, the question is whether the processing of a fraction is componential or holistic. Four experiments employing number comparison tasks were designed to address this issue.

Experiment 1

In Experiment 1, two groups of students with different skills in dealing with fractions were asked to compare fractions' magnitude. In this, as in the following experiments, the distribution of RTs for the different distances from the reference was taken as an index of how fractions are mentally represented. In Experiments 1 and 2, the numerator was always 1. Therefore, the numerical distance between fractions was different from the numerical distance between denominators. For example, the numerical distance between 1/2 and 1/1 (0.5) is about 35 times greater than the distance between 1/8 and 1/9 (0.014), although the difference between denominators is the same.

If the distance effect indexes the nature of the comparison performed, the direction of the SNARC effect indicates how the comparison process is spatially associated to magnitude. What should be noted is that the distance effect and the SNARC effect are independent. If a target is two units smaller than the reference and a second target is two units greater, for example, the distance effect between each of the two targets and the reference would be identical. This identical difference in magnitude, however, would nonetheless produce the SNARC effect, which depends on an asymmetrical association between magnitude and response side.

Method

Participants. There were two groups of participants, one of psychology students (P Group, $n = 10$, 9 women and 1 man, one left-handed, mean age 22.1 years) and one of engineering and physics students (E Group, $n = 10$, all men, one left-handed, mean age 26.6 years). They all reported normal or corrected to normal vision. All participants answered a brief questionnaire to ensure that they were familiar with the notion of fractions. All questions were answered correctly by every participant.

Stimuli. Participants were presented with eight fractions with numerator 1 and denominators varying from 1 to 9 (except for the fraction 1/5, which was the reference). Fractions were presented as two vertically displaced digits separated by a horizontal line and were displayed in white color on a black background. Dimensions were 18 mm in width and 27 mm in height, for 1.7° and 2.6° of visual angle. Viewing distance was about 60 cm.

Procedure. Responses were provided by means of a PST response box (Psychology Software Tools, Inc., Pittsburgh, PA), with response keys 1 and 5 (extreme left and extreme right) pressed with the index fingers of the left and right hand, respectively (distance = 6.7 cm). Half of the participants were asked to press the left key if the fraction was smaller than the reference fraction 1/5 and the right key if the fraction was greater. The other participants had the opposite assignment. In a second block, assignment for the two groups was reversed. Search block had 96 trials, and it was preceded by a brief practice.

A fixation cross was presented centrally on the screen for 600 ms, followed by a blank screen for 1,000 ms. The target fraction was then presented in the center of the screen until response or until 3,000 ms had elapsed. Acoustic feedback was provided in the case of wrong response.

Results and Discussion

Mean error rate was 1.9%. For each participant and fraction medians were calculated on RTs for correct responses.

To investigate the distance effect, all median RTs were entered into a regression model, using either real numerical distance or denominator distance from reference (absolute values) as predictor.

Only the absolute distance of denominator from 5 turned out to be a significant predictor for both groups (P group: $R^2 = .80$; $B = -16.27$, $p < .01$; E group: $R^2 = .61$, $B = -17.42$, $p < .05$; see Figure 1, Panel A). The model with real numerical distance as predictor (numerical value of target fraction minus reference, see Figure 1, Panel B) was not significant for either Group ($p > .1$). The distance effect was thus present only between the denominator of the target and number 5 (reference's denominator).

An analysis of variance (ANOVA) was then used to investigate the association between magnitude and side of response. Note that, when numerical magnitude is task-relevant (Dehaene, Dupoux, & Mehler, 1990; Fischer, 2003; Fischer & Rottmann, 2005), the SNARC effect appears to reflect the target's mere classification as smaller or greater than the reference. Gevers, Verguts, Reynvoet, Caessens, and Fias (2006) referred to this as a "categorical" SNARC effect, in contrast to a "continuous" SNARC effect that is modulated by magnitude and can be observed when numerical magnitude is not relevant for the task (and is best analyzed with a regression analysis).

The ANOVA had Hand (left vs. right), Numerical Magnitude, smaller than 1/5 (1/9, 1/8, 1/7, 1/6) versus larger than 1/5 (1/4, 1/3, 1/2, 1/1), and Group (P Group vs. E Group) as factors. The analysis on error rates (arcsine transformed) did not show any significant effect. The ANOVA performed on median RTs showed a significant interaction Hand × Magnitude interaction, $F(1, 18) = 12.68$, $p < .01$, indicating an association between the response "smaller" and the right hand and between the response "bigger" and the left hand, which is a reversed SNARC effect. Mean RTs for the left hand were slower when responding to small magnitudes compared to large magnitudes, 526 ms versus 480 ms, $F(1, 19) = 10.43$, $p < .01$. In contrast, for the right hand, mean RTs were faster to fractions smaller than the reference compared to fractions

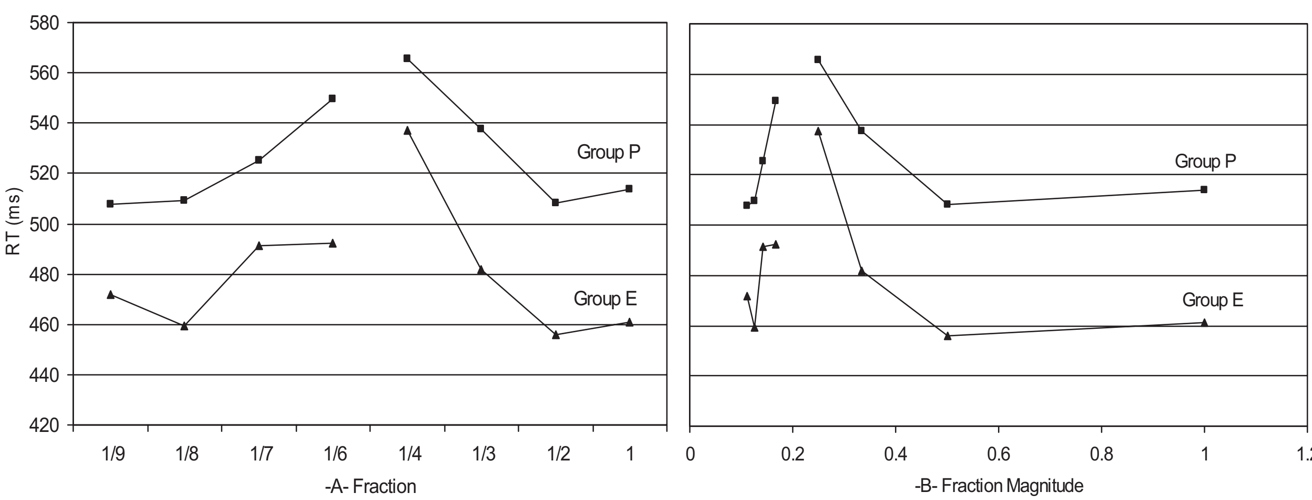

Figure 1. Mean response times (RTs) for each fraction in Experiment 1 as a function of the distance from the reference (1/5 = 0.2). The x-axis in Panel A (left) follows the magnitude of the denominator, whereas, in Panel B (right), it follows the numerical value of the fraction. The latter does not provide a good fit to the data, as shown by the resulting left-skewed function. Separate lines are shown for Group P (psychology students) and Group E (engineering and physics students).

larger than the reference, 479 ms versus 524 ms; $F(1, 19) = 12$, $p < .01$). The main effect of Group was not significant, $F(1, 18) = 1.93$, *ns*, although a trend for the E Group being somewhat faster than the P group can be seen in Figure 2. Notably, the three-way interaction Hand × Magnitude × Group was not significant, $F(1, 18) = 0.650$, *ns*, indicating the absence of differences between the two groups for the SNARC effect. The only significant interaction involving the Group factor was Hand × Group, $F(1, 18) = 4.99$, $p < .05$, which is of no theoretical interest. No other source of variance was significant.

The distance effect between the denominator of the target and the denominator of the reference indicates that a comparison between denominators, rather than between magnitudes, was performed. Moreover, there was an association between left responses and numerically large fractions (with small denominators) and between right responses and numerically small fractions (with large denominators). The fact that a reverse SNARC effect was found indicates that subjects associated the denominator to the left response when it was smaller than the denominator of the reference and to the right response when it was larger, thus indicating an effect of the numerical magnitude of the integers. Therefore, it is clear that the magnitude of the whole fraction was not accessed and not even skilled participants were influenced by the true numerical magnitude of the fraction.

In the following experiment, the format of the reference was changed from the fraction 1/5 to the real number it represents (0.2). The change was aimed at exploring whether using a real number as a reference would favor the access to the true numerical value of the fraction.[1]

Experiment 2

Method

Participants. Eleven psychology students of the University of Padova (2 men, 9 women, all right handed) participated in the experiment. Mean age was 23.9 years.

Stimuli. Stimuli were the same as in Experiment 1.

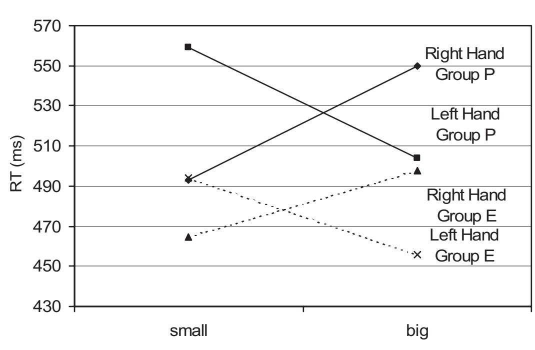

Figure 2. Mean response times (RTs) for right and left hand as a function of fraction magnitude in Experiment 1. The spatial numerical association of response codes (SNARC) effect is reversed compared to the usual association between magnitude and response side. Separate lines are shown for Group P (psychology students) and Group E (engineering, physics, and computer science students).

Procedure. Participants were asked to compare the target fraction presented with the numerical reference 0.2 (smaller or larger). For the other details, the procedure was the same as in Experiment 1.

Results and Discussion

Errors were, on average, 3.6%, after the exclusion of one participant with 25% errors. Data treatment was the same as for Experiment 1. As for Experiment 1, the distance effect was evaluated through regression analyses. The true numerical distance between the fraction and the reference was not a significant predictor of the RTs ($p > .1$). In contrast, the distance between the denominator of the target and number 5 turned out to be highly significant in the regression analysis (in absolute value; $R^2 = .79$, $B = -15$, $p < .01$; see Figure 3). Number 5 is the denominator's value for a fraction with numerator 1 and magnitude 0.2.

The ANOVA on RTs, using Hand (left vs. right) and Numerical Magnitude, smaller than 1/5 (1/9, 1/8, 1/7, 1/6) versus larger than 1/5 (1/4, 1/3, 1/2, 1/1) as factors, resulted in a nonsignificant interaction Hand × Magnitude, $F(1, 9) = .72$, *ns*, indicating the absence of the SNARC effect. The same analysis performed on error rates (arcsine transformed) was not significant either, $F(1, 9) = 1.13$, *ns*. A one-way ANOVA comparing RTs of Experiment 1 (Group P) with those of Experiment 2 showed no difference, $F(1, 18) = .21$, *ns*, and the same was true of an identical ANOVA on error rate $F(1, 18) = 2.46$, $p = .13$.

Providing a real number as reference should have facilitated accessing the numerical value of the fractions. The real numerical distance between the fraction and the reference, in fact, did not fit the data. The most likely hypothesis concerning the procedure adopted by participants is that they transcoded the reference into 1/5 and then processed the denominator only, as had happened in Experiment 1. This is clearly shown by the presence of a distance effect between the denominator and number 5. Note that transcoding of the reference 0.2 into the corresponding fraction (1/5) could take place at each trial or only once at the very start of the experiment. The latter strategy would be more effective, because keeping the reference unchanged throughout the task permits the avoidance of a potential RT cost for each trial caused by the additional transcoding operation. Indeed, RTs did not differ between Experiment 1 (Group P) and Experiment 2.

In summary, the results of Experiment 2 show that participants adopted the strategy of transcoding the reference (a real number) into a fraction and then systematically used its denominator for the comparison with the denominator of the target fractions. Employing this strategy no doubt required a series of operations, but this procedure was preferred to that of having recourse to direct activation of numerical magnitude. The lack of a significant SNARC effect is difficult to interpret. Considering that the SNARC effect was present and significant in Experiment 1, which yielded very similar results to Experiment 2, its absence in the latter is likely attributable to a type-2 error.

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[1] We are grateful to Veronique Izard for this suggestion.

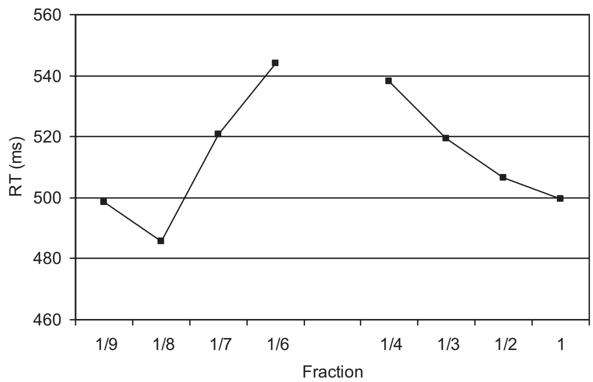

Figure 3. Mean response times (RTs) for each fraction in Experiment 2 as a function of the distance from the reference number (0.2 = 1/5). The scale of the x-axis is based on denominator magnitude.

Experiment 3

The use of componential processing in Experiments 1 and 2 could reflect an effective use of strategies that come about during childhood as a result of learning to use fractions. Indeed, strategy development is an important component of arithmetic learning (Lemaire & Siegler, 1995). Alternatively, strategies could be flexibly adapted to the specific context to circumvent genuine difficulties in accessing numerical magnitude. Experiment 3 was designed to further investigate this issue, using a wider range of fractions. By varying the numerator and the denominator, participants were compelled to process both operands in order to respond correctly.

Method

Participants. Sixteen psychology students of the University of Padova (4 men and 12 women, all right handed) participated in the experiment. Mean age was 23.7 years.

Stimuli. Participants were presented with fractions with numerators from 1 to 9 and denominators from 4 to 6, excluding fractions with the numerator equal to the denominator: 4/4, 5/5, and 6/6. Thus, there were 24 different fractions. Stimulus width and height were the same as in Experiments 1 and 2.

Procedure. Stimulus presentation was the same as in the previous experiments. Participants were asked to press the assigned response key if the fraction presented was smaller or greater than 1. Every participant responded to two blocks with 120 trials each, switching the response key assignment at the end of each block.

Results and Discussion

Errors were, on average, 5%. As for the previous experiments, the distance effect was evaluated through regression analysis on RTs. The real numerical distance between target fraction and reference (1) and the distance between numerator and denominator were used as predictors (absolute values) in separate regression analyses (see Figure 4). The absolute distance between numerator and denominator best fitted RTs ($R^2 = .54$, $B = -42.1$, $p < .001$). The real numerical distance was significant too, but the regression model showed a poorer fit to the data ($R^2 = .49$, $B = -179.9$, $p < .001$). Note that the real numerical distance is highly correlated to the distance between numerator and denominator ($r = .93$, $p < .001$). By inspecting Figure 4A, however, it is apparent that the numerator 5 acted as a reference point for all fractions. The overlap for the three types of fractions (i.e., n/4, n/5, and n/6) suggests that 5/5, which lies at the middle of the range of the presented fractions, was used as reference. This could reflect the strategy of transcoding the reference 1 into 5/5. Indeed, the fit to the RTs increases substantially if the numerical distance between the fraction numerator and number 5 is used as a predictor ($R^2 = .89$, $B = -54$, $p < .001$). This finding clearly shows that participants did not access the real numerical magnitude of the fractions.

To test the presence of the SNARC effect, data were analyzed with an ANOVA with Hand (left vs. right) and Numerical Magnitude (smaller vs. larger than 1) as factors. There was a significant Hand × Magnitude interaction, $F(1, 15) = 10.61$, $p < .01$, indicating a regular SNARC effect attributable to the magnitude of the fractions (see Figure 5). Mean RTs for the right hand were 601 ms when responding to fractions larger than 1 and 665 ms when

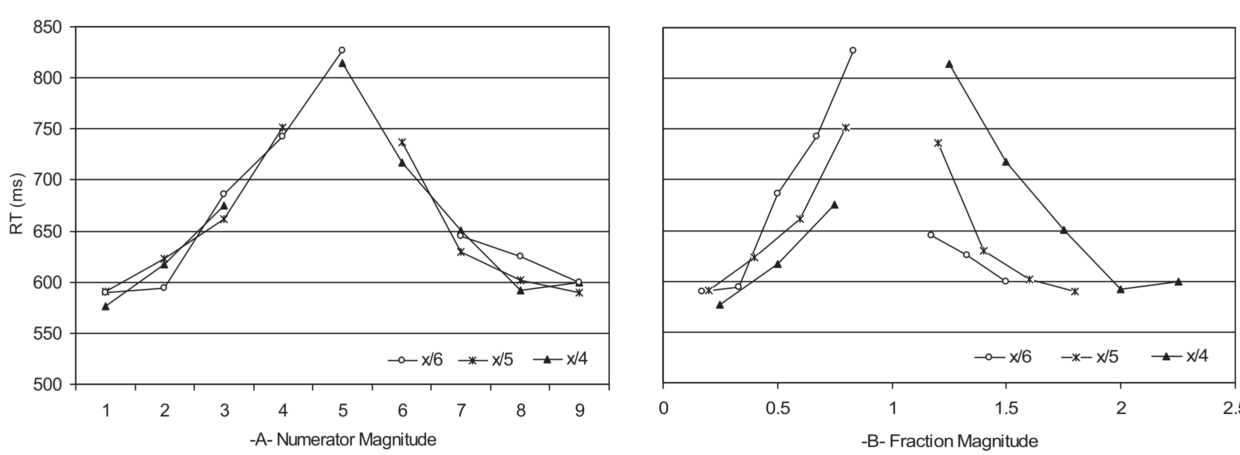

Figure 4. Mean response times (RTs) for each fraction in Experiment 3 as a function of the distance from the reference number (1). The x-axis in Panel A (left) shows the magnitude of the numerator; the x-axis in Panel B (right) shows the fraction magnitude.

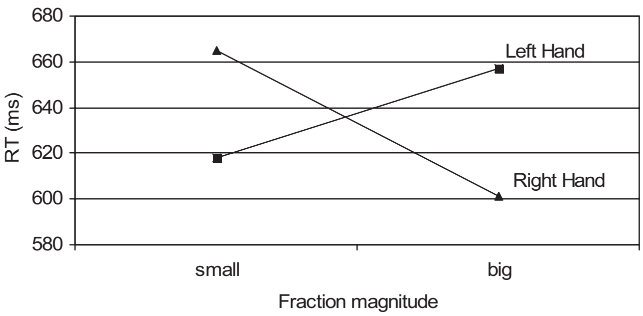

Figure 5. The Hand × Magnitude interaction in Experiment 3 indicates a regular spatial numerical association of response codes (SNARC) effect.

responding to fractions smaller than 1, $F(1, 15) = 7.76$, $p < .05$. Also for the left hand, mean RTs for fractions smaller than 1 and for fractions larger than 1 differed significantly, 618 ms vs. 657 ms, $F(1, 15) = 7.87$, $p < .05$. The same ANOVA performed on error rates (arcsine transformed) confirmed the significant Hand × Magnitude interaction, $F(1, 15) = 6.29$, $p < .05$.

Compared to the effect obtained in Experiment 1, the SNARC effect was opposite in direction. The association found in Experiment 3 was between large numbers and right responses and between small numbers and left responses, as usually reported for the SNARC effect. However, this result is not surprising because the numerator's magnitude, on which the strategy rested, was congruent with the numerical magnitude of the fraction. In Experiments 1 and 2, instead, the strategy was applied to the denominator: When the denominator increased there was a decrease in the numerical magnitude of the fraction.

Results suggest that processing both numerator and denominator, which was mandatory to perform the task, still failed to produce the activation of the numerical magnitude of the fraction. After establishing if the numerator was larger or smaller than the denominator, participants did not access the numerical magnitude, as clearly demonstrated by the fact that the distance effect was determined by distance between numerator and 5 rather than by the fraction's magnitude. Comparing the numerator with the denominator did not favor access to the ratio indicated by the whole fraction.

Experiment 4

The results of the experiments reported above suggest that participants adopt strategies that are tied to the specific experimental context. The flexible use of strategies is compatible with the hypothesis that participants avoid, if they can, the use of the real numerical value of the fraction but resort instead to their integer components to solve the task at hand. Experiment 4 was designed to further explore this issue. Participants might adjust their strategies either to the interval of target fractions, to the reference number, or to both. We therefore mixed the references and the intervals adopted in the previous experiments. Specifically, we used a variable standard design in which the reference was either 1/5 or 1. The type of target fraction was x/5 or 1/x, respectively.

It is important to note that the variable standard design should strongly discourage participants from using strategies based on componential processing. Indeed, the trial-by-trial change of reference would render the use of strategies more costly and should favor instead the use of real numerical values (i.e., holistic processing). We also tested the participants on a separate test of fraction knowledge to ensure that at least some of them would be highly skilled in using fractions.

Method

Participants. Twenty-four students of the University of Padova (9 women, 15 men; 3 left-handed) participated in the experiment. Mean age was 23.8 years. Of the participants, 14 were psychology students and 10 were students of engineering, physics, or computer science.

Stimuli. They were fractions with numerator 1 and denominator from 1 to 9 (excluding 1/5, as in Experiments 1 and 2) or fractions with numerator from 1 to 9 and denominator 5 (excluding 5/5, as the central interval used in Experiment 3). Stimulus width and height were the same as in Experiments 1, 2, and 3.

Procedure. Participants' expertise with fractions was assessed through a paper-and-pencil test that contained 10 operations and 10 magnitude comparisons (see Appendix A). Problems were printed on a single A4 sheet and participants were asked to write the solution next to each problem. They were allowed to calculate the result in written form, if needed, without time limit.

The computer-based experiment was then administered. Each trial started with a fixation cross, followed by presentation of a target fraction. The reference for the numerical comparison, either 1/5 or 1, was randomly selected for each trial. The identity of the reference was signaled through a color cue: Participants were instructed to compare the target fraction with 1/5 if the fixation cross was green and with 1 if the fixation cross was red. Unbeknown to the participants, the reference was always 1/5 for fractions with numerator 1 (as in Experiment 1), whereas the reference was the unit (1) for fractions with denominator 5 (as in Experiment 3). Fractions were presented until a response was given. No feedback was provided. Every participant responded to two blocks with 128 trials each, switching the response key assignment at the end of the first block.

Results and Discussion

Four participants were excluded from the analyses because of their very high error rates in the experimental task (from 16.8% to 28.5%). The remaining 20 participants were divided into two groups on the basis of their performance in the paper-and-pencil test on fractions. Highly skilled participants, whose accuracy was perfect (i.e., 0 errors), were placed in Group 1 ($n = 9$). The other participants ($n = 11$) showed a lower accuracy ($M = 1.7$ errors, range 1–3 errors) and were placed in Group 2.

RTs over 3,000 ms (fewer than 2.5%) were discarded from analysis. The overall mean (from medians) RT for correct responses was 864 ms. An ANOVA on RTs with Hand (left vs. right), Numerical Magnitude (smaller vs. larger), Reference (1/5 vs. 1), and Group (1 vs. 2) as factors showed significant main effects of Reference $F(1, 18) = 27.81$, $p < .001$ (800 ms for reference 1 vs. 929 ms for reference 1/5) and Magnitude $F(1, 18) = 11.12$, $p < .01$. This was qualified by a significant Reference × Magnitude interaction, $F(1, 18) = 41.02$, $p < .001$. Larger magnitudes were slower than smaller ones for reference 1/5, 962 ms vs. 896 ms; $F(1, 19) = 11.2$, $p < .01$, whereas the opposite was true for reference 1, 723 ms vs. 876 ms; $F(1, 19) = 49.79$, $p < .001$. The interaction Hand × Magnitude was not significant, $F(1, 18) = .38$, $p = .55$. The effect of Group was not significant either, $F(1, 18) = .022$, $p = .885$, nor was any interaction with other factors.

The same ANOVA performed on error rates (arcsine transformed) showed a main effect of Reference, $F(1, 18) = 6.15$, $p < .05$ (5.1% for reference 1/5 and 2.7% for reference 1) and a significant interaction Reference × Magnitude, $F(1, 19) = 24.88$, $p < .001$. The latter paralleled the pattern of RTs: Smaller magnitudes were easier than larger ones for reference 1/5 (3% vs. 7.3%), whereas the opposite was true for reference 1 (3.7% vs. 1.6%). All other main effects and interactions were not significant, including the two way interaction Hand × Magnitude, $F(1, 18) = .68$, *ns*.

The interaction between Reference and Magnitude that we found in both RTs and accuracy would seem puzzling at first sight. Note, however, that the trials yielding faster and more accurate responses are those in which a comparison of the target fraction to either of the references would produce the same result. That is, fractions 1/9, 1/8, 1/7 and 1/6 are smaller than both 1/5 and 1, whereas fractions 6/5, 7/5, 8/5 and 9/5 are larger than both 1/5 and 1. In contrast, the other half of trials can be considered incongruent, because a comparison with either of the references yields opposite responses. For instance, 1/4 is larger than 1/5 but it is smaller than 1; similarly, 4/5 is larger than 1/5 but is smaller than 1.

The reference-congruency effect revealed by the interaction between reference and magnitude shows once again a strategic adaptation to the task that effectively circumvents the problem of comparing the real numerical magnitudes of the fractions. Therefore, the following analyses are focused on the incongruent trials, those in which the target fraction had to be compared to the relevant reference to produce a correct response. These are the fractions larger than 1/5 for the 1/x type (i.e., 1/4, 1/3, 1/2, 1/1) and the fractions smaller than 1 for the x/5 type (i.e., 1/5, 2/5, 3/5, 4/5). Interestingly, a comparison between the two types of fraction (only incongruent trials) showed that 1/x fractions are responded to more slowly than x/5 fractions, $F(1, 19) = 9.22$, $p < .01$. This difference is likely to reflect the fact that the former require a "larger" response even though they have a small denominator (1 to 4), that is, a SNARC-like effect. Note that reliance on the denominator for 1/x fractions was also found in Experiment 1.

Inspection of the data (see Figure 6) clearly suggests that incongruent trials were responded to through a comparison of the variable part of the fraction with the informative part of the reference. For fractions of the x/5 type the comparison was between the numerator x and the reference 1. In contrast, for fractions of the 1/x type the comparison was between the denominator x and the denominator of the reference, that is 5. Indeed, the real numerical distance between target fraction and reference predicts, at least for fractions of the x/5 type, an effect that goes in the opposite direction from what is observed. That is, an effect of the real numerical distance (Figure 6B) should produce longer RTs as the fraction value gets closer to that of the reference (i.e., a positive slope), but the observed negative slope (Figure 6A) can be explained by the fact that the distance between the numerator and 1 gets smaller. To confirm that the slope was negative, we computed for each subject the slope of the regression equation using the method recommended by Lorch and Myers (1990). Indeed, the

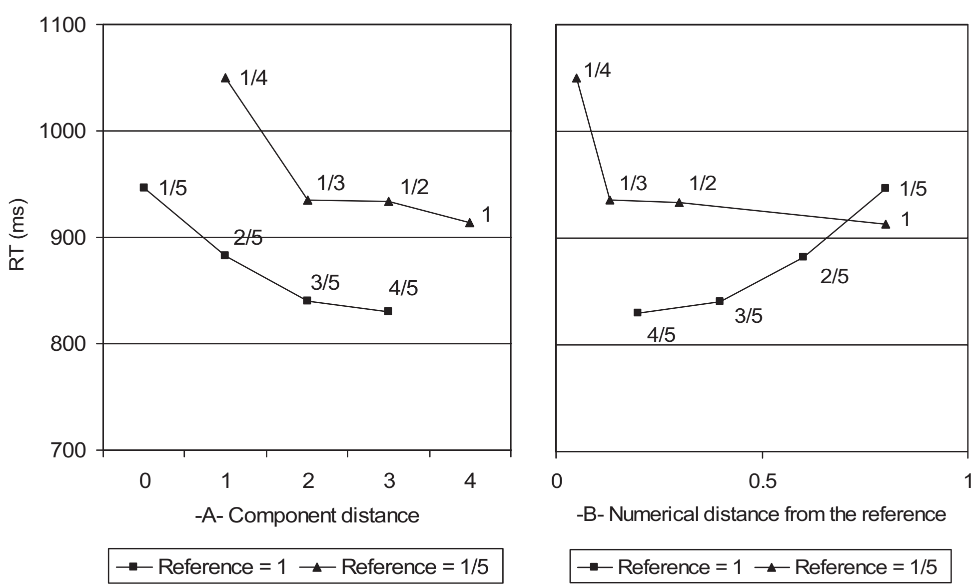

Figure 6. Distance effect in Experiment 4 (incongruent trials only). The reference was 1 for x/5 fractions and 1/5 for 1/x fractions. In Panel A (left), the x-axis shows the distance between the variable part of the fraction and the informative part of the reference. For the x/5 type (squares) these are the numerator of target and the reference itself (1). For the 1/x type (triangles), these are the denominator of the target and the denominator of the reference (5). In Panel B (right), the x-axis shows the real numerical distance between target and fraction magnitude.

slopes were significantly negative in a t test against zero, $t(19) = -2.76$, $p < .05$, two-tailed. The same pattern was found for the 1/x fractions: the individual regressions, using the distance of the denominator from 5 as predictor, revealed a significant negative slope, $t(19) = -2.36$, $p < .05$, two-tailed.

In summary, the results of Experiment 4 suggest that participants flexibly employed different strategies that changed according to the type of fraction. They performed a comparison only when it was necessary to provide a correct answer (i.e., in the case of incongruent trials) and focused only on the informative (variable) integer component of the fraction. What is more important, however, is that the increased complexity of the task did not discourage participants from using strategies based on componential processing. Even a trial-by-trial change of reference did not elicit the use of real numerical values. This was true regardless of the participants' skill and familiarity with fractions.

General Discussion

The present study demonstrated that classical effects found in numerical cognition, like the distance effect and the SNARC effect, can be used to investigate the mental representation of fractions. In particular, we investigated the processing of the fraction's magnitude, showing that the numerical value of a fraction was not accessed even in the case of skilled and highly educated adult participants. A componential processing of the fraction was performed in place of the activation of its numerical magnitude, although that required transforming the reference, as happened in Experiment 2, or employing complex strategies, as in Experiments 3 and 4. The magnitudes of the operands of a fraction, instead, were separately and automatically accessed.

The componential processing of a fraction, which implies the activation of the magnitude of the single digits forming it, can explain the difficulties in accessing and understanding its true numerical value. For example, children are able to multiply fractions without difficulties by simply applying learned rules, but they rarely understand the meaning behind the computation and have "difficulties in generalizing the information to other situations, especially when facing complex problems" (Wu, 2001, p. 174).

Componential processing is also found in other number processing tasks that are routinely performed by adults. When the magnitudes of two-digit numbers are compared, responses are slower if, for example, a number is greater than the other number but has a smaller unit value (unit–decade incompatibility; Nuerk, Weger, & Willmes, 2001). The use of negative numbers provides further evidence of componential processing. A reverse SNARC effect was observed in a parity judgment task with negative numbers (Fischer & Rottmann, 2005), suggesting that participants responded to the magnitude of the digit without processing the minus sign.

Our study, the first to investigate the mental representation of fractions in adults, shows that skilled participants prefer to have recourse to heuristics based on integer numbers and do not automatically access the real number that the fraction represents. The use of strategies can reflect learning processes (Lemaire & Siegler, 1995); but, in the present case, it is apparently a way to avoid accessing the numerical magnitude of the fraction. Notably, the heuristics adopted by our participants were tied to the specific experimental context and thus must have been set up on the fly. When the task became more challenging, as in Experiment 4, new and more complex strategies were adopted even though it was arguably costly to do so.

The prevalence of integer number over real magnitudes in processing fractions has been widely described in developmental and instructional studies of children, and it has been referred to as whole number bias (Ni & Zhou, 2005, for review). Hartnett and Gelman (1998) suggested that magnitude processing is overcome by counting processes, not isomorphic with magnitude, and related only to integer numbers. Children acquire a conceptual understanding of fraction and rational number as an interrelated body of representations, including representations of division and density of number (Smith et al., 2005). In contrast to a widely held belief, however, it appears that the whole number bias is largely carried on into adulthood. Skilled adults have acquired the concept of rational number and the mastery of procedures for operating with fractions (cf. the paper-and-pencil test in Experiment 4), but this is not mirrored by a significant change in how the magnitude of fractions is mentally represented. Indeed, the main difference between children and adults seems to reside in the ability of the latter to circumvent the challenge posed by fractions through a flexible use of strategies.

Overall, these results are at odds with the hypothesis that the representation of numbers is rooted in an innate system for approximate magnitudes (that is, real numbers; Gallistel and Gelman, 2000), which would instead have predicted an easy access to the true fraction's value. In effect, determining the value of a fraction can be considered as estimating the result of a division. Gallistel and Gelman, commenting on the developmental studies of fraction knowledge, noted this paradox: "If humans represent numerosities in terms of magnitudes, why do they have so much trouble learning the mathematical conception of rational numbers (mastering fractions)?" (p. 64). Nonetheless, Gallistel and Gelman claim that numerical reasoning operates on real numbers and that "getting from integers back to the real numbers has been the work of man" (p. 65). Notably, mathematicians like Kronecker had the opposite intuition (Bell, 1937).

It may be argued that the difficulty in accessing the real magnitude of fractions can be explained in terms of the difficulty of peripheral transcoding processing necessary to translate the fraction to an internal mental representation of quantity. Indeed, some have suggested that children's difficulty with fractions and rational numbers might also be caused by the confusion of using the same written symbols for both whole numbers and fractional amounts (e.g., Mix et al., 1999; Sophian, Garyantes, & Chang, 1997). This explanation, however, is hardly tenable in the case of skilled adults: Why would instruction and extensive practice fail in establishing an effective mapping between fractions and real magnitude?

The strong tendency shown by participants to use strategies based on the manipulation of the integer components better fits with the hypothesis that the processing of symbolic numbers is rooted in the representation of discrete, exact numerosities (Zorzi & Butterworth, 1999; Zorzi, Stoianov, & Umiltà, 2005; Zorzi et al., 2006). The whole number bias in adults is also consistent with the hypothesis that the adult numerical system for exact numbers is bootstrapped, through the use of the set of counting words, from the small number system (which is accurate for numbers up to 3 and is essentially the perceptual system for tracking objects) rather than from the approximate number system (for a review of the core systems of number, see Feigenson, Dehaene & Spelke, 2004). It should be noted that neither interpretation denies the existence of a phylogenetically distinct system for approximate magnitudes, which is clearly involved in processing and manipulating nonsymbolic stimuli, like collections of dots (e.g., Barth et al., 2006; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004). Nevertheless, a system for representing exact (integer) numbers seems to be preferred to the use of (approximate) analogue magnitudes even when dealing with fractions, a system of symbols that formally correspond to real numbers.

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Appendix A

Problems Used in the Paper-and-Pencil Test (Experiment 4)

$1/4 + 3/8 =$

$3/5 - 1/3 =$

$3/10 + 5/6 =$

$5/8 - 3/4 =$

$3/4 \times 1/6 =$

$25/3 \times 7/10 =$

$1/2 \div 3/4 =$

$5/6 \div 21/3 =$

$(1/2)(2/3) + (5/6)(2/5) =$

$(1/4)(3 - 3/5) =$

$1/1 > 1/5$ True – False

$1/4 > 1/5$ True – False

$1/6 > 1/5$ True – False

$1/8 > 1/5$ True – False

$1/5 > 1$ True – False

$3/5 > 1$ True – False

$7/5 > 1$ True – False

$3/7 > 3/9$ True – False

$8/6 > 6/4$ True – False

$7/8 > 2/3$ True – False

Received March 28, 2006
Revision received January 8, 2007
Accepted February 25, 2007 ■